

Random integral representations for free-infinitely divisible and tempered stable distributions*

Zbigniew J. Jurek

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ABSTRACT. There are given sufficient conditions under which mixtures of dilations of Lévy spectral measures, on a Hilbert space, are Lévy measures again. We introduce some random integrals with respect to infinite dimensional Lévy processes, which in turn give some integral mappings. New classes (convolution semigroups) are introduced. One of them gives an unexpected relation between the free (Voiculescu) and the classical Lévy-Khintchine formulae while the second one coincides with tempered stable measures (Mantegna nad Stanley) arisen in statistical physics.

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In probability theory and mathematical statistics the Fourier and Laplace transforms are the main tools used to prove weak limit theorems or to identify probability distributions. These are purely *analytic methods* used in other branches of mathematics as well. Jurek and Vervaat (1983) had introduced *the method of random integral representations* that allows to describe distributions as laws of some random integrals with respect to Lévy processes. In fact, this method was introduced for a study of the class L of selfdecomposable probability measures on Banach spaces. Namely, it was proved that for a probability measure μ , on a Banach space E , we have

$$\mu \in L \text{ iff } \mu = \mathcal{L}\left(\int_0^\infty e^{-t} dY(t)\right) \text{ for a unique Lévy process } Y. \quad (1)$$

More precisely, to insure the existence of the above improper random integrals one needs that $\mathbb{E}[\log(1+||Y(1)||)] < \infty$, and then the induced random integral mapping \mathcal{I}

$$ID_{\log}(E) \ni \mathcal{L}(Y(1)) \rightarrow \mathcal{L}\left(\int_0^\infty e^{-t} dY(t)\right) \in L(E)$$

is an isomorphism between the corresponding convolution semigroups of probability measures: infinitely divisible with finite logarithmic moments ID_{\log} and selfdecomposable ones L . To the (unique) Lévy process Y in (1) one refers as the *background driving Lévy process* for the selfdecomposable measure μ , in short its BDLP. Besides "randomness" the *stochastic method* given by the integral representation (1), gives easily characterization of measures in terms of their Fourier transforms. Moreover, it allows to incorporate space and time changes in order to retrieve the *BDLP*, i.e.,

$$\mathcal{L}\left(\int_0^c e^{-t} dY(t/c)\right) \Rightarrow \mathcal{L}(Y(1)), \text{ as } c \rightarrow 0.$$

The aim of this note is to extend the random integral representation approach to more general schemes than those in Jurek (1982, 1983, 1985, 1988) and Jurek and Vervat (1983). As a consequence we will find such representations for two classes (semigroups) of measures. First, a class \mathcal{E} that coincides with a class of free infinitely divisible measures in Voiculescu sense (cf. Barndorff-Nielsen and Thorbjørnsen (2002)) and the second, \mathcal{TS} coincides with the tempered stable distributions (these are a generalization of the titled stable measures) that arose among others in statistical physics; (cf.

Mantegna and Stanley (1994), Novikov (1995), Kaponen (1995) and Rosiński (2002), (2004)). One may expect that the random integral representations will help in some simulation problems.

Our results are mainly given in a generality of measures on a Hilbert space, with some digression to a case of Banach spaces. However, methods of proofs are dimensionless so they can be read for Euclidean spaces as well.

1. Introduction and terminology. Let E denotes a real separable Banach space, E' its conjugate space, $\langle \cdot, \cdot \rangle$ the usual pairing between E and E' , (this is just a scalar product in a case when E is a Hilbert or an Euclidean space), and $\|\cdot\|$ the norm on E . The σ -field of all Borel subsets of E is denoted by \mathcal{B} , while \mathcal{B}_0 denotes Borel subsets of $E \setminus \{0\}$. By $\mathcal{P}(E)$ or simply by \mathcal{P} , we denote the (topological) semigroup of all Borel probability measures on E , with convolution “ $*$ ” and the topology of weak convergence “ \Rightarrow .”

Recall that a measure $\mu \in \mathcal{P}$ is called *infinitely divisible* if for each natural $n \geq 2$ there exists $\nu_n \in \mathcal{P}$ such that $\nu_n^{*n} = \mu$. The class $ID(E)$, of all infinitely divisible probability measures on E , is a closed topological subsemigroup of \mathcal{P} . Each ID distribution μ is uniquely determined by a triple: a shift vector $a \in E$, a Gaussian covariance operator R , and a Lévy spectral measure M ; we will write $\mu = [a, R, M]$. These are the parameters in the Lévy-Khintchine representation of the characteristic function $\hat{\mu}$, namely

$$\begin{aligned} \mu \in ID \quad \text{iff} \quad \hat{\mu}(y) &= \exp[\Phi(y)], \quad \text{where} \\ \Phi(y) &:= i \langle y, a \rangle - 1/2 \langle Ry, y \rangle + \\ &\quad \int_{E \setminus \{0\}} [e^{i \langle y, x \rangle} - 1 - i \langle y, x \rangle 1_{\|x\| \leq 1}(x)] M(dx), \quad y \in E'; \end{aligned} \quad (2)$$

Φ is called the *Lévy exponent* of $\hat{\mu}$ (cf. Araujo and Giné (1980), Section 3.6). For $\mu \in ID(E)$ one can define arbitrary positive convolution powers, namely

$$\text{if } \mu = [a, R, M] \in ID(E) \text{ then } \mu^{*c} = [c \cdot a, c \cdot R, c \cdot M], \text{ for any } c \geq 0, \quad (3)$$

and

$$T_a \mu = [a_c, c^2 R, T_c M], \text{ where } a_c = c a + c \int_E x [1_B(cx) - 1_B(x)] M(dx) \quad (4)$$

where the mapping T_c is given by $T_c x = cx, x \in E$ and $T_c M(A) = M(c^{-1}A)$ for all $A \in \mathcal{B}_0$.

In a case when ρ is the probability distribution of an E -valued random element ξ then $T_a\rho$ is the probability distribution of the random element $a \cdot \xi$, for any real number a .

Let $\mathcal{M}(E)$ denotes the totality of all Lévy spectral measures on a space E . It is a positive cone but it is also closed under dilations T_a , i.e.,

$$M \in \mathcal{M}(E) \text{ iff } T_a M \in \mathcal{M}(E), \text{ for all } a \in \mathbb{R}. \quad (5)$$

Hence we conclude that

$$M \in \mathcal{M}(E) \text{ iff } \sum_{i=1}^k c_i \cdot T_{a_i} M \in \mathcal{M}(E) \text{ for all } k \geq 1, c_i > 0, a_i \in \mathbb{R}. \quad (6)$$

2. The λ -mixtures of Lévy spectral measures. Our main objective in this section is to provide a method of constructing Lévy spectral measures using the mixtures of $T_t M$. Namely, for a non-negative Borel measure λ on $\mathbb{R}^+ = (0, \infty)$ and a Borel measure M on $E \setminus \{0\}$ we define

$$M^{(\lambda)}(A) := \int_0^\infty (T_t M)(A) \lambda(dt) = \int_0^\infty \int_{E \setminus \{0\}} 1_A(tx) M(dx) \lambda(dt), \quad A \in \mathcal{B}_0, \quad (7)$$

and call it the λ -mixture of $T_t M, t > 0$. Questions on mixtures of Lévy spectral measures on Banach spaces were investigated in Jurek (1990). In particular, it was proved that

$$\begin{aligned} \text{if } M^{(\lambda)} \in \mathcal{M}(E) \text{ then } M \in \mathcal{M}(E), \quad \int_0^\infty \min(1, t^2) \lambda(dt) < \infty \text{ and} \\ \int_0^\infty M(\{x : \|x\| > t^{-1}\}) \lambda(dt) = \int_{E \setminus \{0\}} \lambda(\{s : s > \|x\|^{-1}\}) M(dx) < \infty; \end{aligned} \quad (8)$$

cf. Jurek (1990), Proposition 2. The converse implication to (8) is not completely settled. Note that for measures λ with finite support and arbitrary Lévy spectral measure $M \in \mathcal{M}(E)$, (6) gives that $M^{(\lambda)} \in \mathcal{M}(E)$. Here is an extension for more general λ but less general M .

PROPOSITION 1. *Suppose that λ and M are Borel measures on $(0, \infty)$ and E respectively, such that*

$$\begin{aligned} \int_{E \setminus \{0\}} [||x|| \int_0^{||x||^{-1}} t \lambda(dt) + \lambda(\{s : s > ||x||^{-1}\})] M(dx) = \\ \int_0^\infty [t \int_{\{0 < ||x|| \leq t^{-1}\}} ||x|| M(dx) + M(\{||x|| > t^{-1}\})] \lambda(dt) < \infty. \end{aligned} \quad (9)$$

Then $M^{(\lambda)}$ and M are Lévy spectral measures on E and λ is a Lévy spectral measure on $(0, \infty)$.

Proof. From Araujo-Gine (1980), Theorem 6.3, we know that if a measure M integrates $\min(1, ||x||)$ on a Banach space E then it is a Lévy spectral measure. Condition (9) is just that integrability condition for $M^{(\lambda)}$. [It was written in a form of a sum of two integrals to indicate two different behaviors of Lévy measures: on open neighborhoods of zero and their complements.]

Finally, from the inequality $\min(1, t) \cdot \min(1, ||x||) \leq \min(1, t||x||)$, for all $t \geq 0$ and $x \in E$, and formula (8) we infer the remaining claims.

PROPOSITION 2. *Suppose that H is a real separable Hilbert space, and M and λ are Borel measures on H and on $(0, \infty)$, respectively. Then $M^{(\lambda)}$ is a Lévy spectral measure if and only if*

$$\begin{aligned} \int_{H \setminus \{0\}} [||x||^2 \int_0^{||x||^{-1}} t^2 \lambda(dt) + \lambda(\{s : s > ||x||^{-1}\})] M(dx) = \\ \int_0^\infty [t^2 \int_{\{0 < ||x|| \leq t^{-1}\}} ||x||^2 M(dx) + M(\{||x|| > t^{-1}\})] \lambda(dt) < \infty, \end{aligned} \quad (10)$$

Moreover, M is a Lévy spectral measures on H and λ is a Lévy spectral measure on \mathbb{R} .

Proof. In a case of Hilbert space, for a measure M to be a Lévy spectral measure it is necessary and sufficient that M integrates $\min(1, ||x||^2)$; cf. Parthasarathy (1967), Chapter VI, Theorem 4.10. The condition (10) is just the mentioned integrability condition for $M^{(\lambda)}$. Furthermore, as before we use the inequality $\min(1, t^2) \cdot \min(1, ||x||^2) \leq \min(1, t^2 ||x||^2)$, for all $t \geq 0$ and $x \in H$, and the proof is complete.

Examples. A). Let e denotes the standard exponential distribution with the density $e^{-t}, t > 0$. Then

COROLLARY 1. *On any real separable Hilbert space H we have that*

$$M^{(e)} \text{ is a Lévy spectral measure iff so is } M. \quad (11)$$

Proof. Let M be a Lévy spectral measure on H . Since we have that

$$\begin{aligned} g(\|x\|) &:= \|x\|^2 \int_0^{\|x\|^{-1}} t^2 e^{-t} dt + \int_{\|x\|^{-1}}^{\infty} e^{-t} dt = \\ &2\|x\|^2 [1 - e^{-\|x\|^{-1}} (1 + \|x\|^{-1})], \end{aligned}$$

therefore $g(\|x\|) \leq 2\|x\|^2$, for $\|x\| \leq 1$. On the other hand, using the power series expansion we also get $\lim_{\|x\| \rightarrow \infty} g(\|x\|) = 1$, which implies that $g(\|x\|) \leq K$, for $\|x\| > 1$. Consequently,

$$\begin{aligned} \int_{H \setminus \{0\}} \min(1, \|x\|^2) M^{(e)}(dx) &= \int_{\{0 < \|x\| \leq 1\}} g(\|x\|) M(dx) \\ &+ \int_{\{\|x\| > 1\}} g(\|x\|) M(dx) \leq \\ &2 \int_{\{0 < \|x\| \leq 1\}} \|x\|^2 M(dx) + K \int_{\{\|x\| > 1\}} M(dx) < \infty \end{aligned}$$

and Proposition 2 gives that $M^{(e)}$ is spectral measure.

Converse implication also follows from Proposition 2, and thus the proof is complete.

B). Let $\rho_\alpha(dt) := t^{-\alpha-1} e^{-t} dt$ be a measure on $(0, \infty)$.

COROLLARY 2. *On any Hilbert space H , if $M^{(\rho_\alpha)}$ is a Lévy spectral measure then so is M and*

$$\begin{aligned} 0 < \alpha < 2, \int_{\|x\| > 1} \|x\|^\alpha M(dx) &< \infty, \\ \text{and for all } s > 0, \int_{\{0 < \|x\| \leq 1\}} \|x\|^\alpha e^{-s/\|x\|} M(dx) &< \infty. \end{aligned}$$

Conversely, if $0 < \alpha < 2$ and $\int_{\{\|x\| > 0\}} \|x\|^\alpha M(dx) < \infty$ then $M^{(\rho_\alpha)}$ and M are Lévy spectral measures.

Proof. Let us introduce a function

$$\begin{aligned} h_\alpha(|x|) &:= |x|^2 \int_0^{|x|^{-1}} t^{(2-\alpha)-1} e^{-t} dt + \int_{|x|^{-1}}^\infty t^{-\alpha-1} e^{-t} dt \\ &= |x|^\alpha \left[\int_0^1 t^{(2-\alpha)-1} e^{-t/|x|} dt + \int_1^\infty t^{-\alpha-1} e^{-t/|x|} dt \right]. \end{aligned}$$

From Proposition 2, we have that $\int_{H \setminus \{0\}} h_\alpha(|x|) M(dx) < \infty$. Consequently, $h_\alpha(|x|) < \infty$ for M -a.a.x. Consequently, $2-\alpha > 0$ (from the first integral) and $\alpha > 0$. Since we also have that

$$\begin{aligned} \int_{H \setminus \{0\}} h_\alpha(|x|) M(dx) &= \int_0^1 t^{(2-\alpha)-1} \left(\int_{H \setminus \{0\}} |x|^\alpha e^{-t/|x|} M(dx) \right) dt \\ &\quad + \int_1^\infty t^{-\alpha-1} \left(\int_{H \setminus \{0\}} |x|^\alpha e^{-t/|x|} M(dx) \right) dt < \infty. \quad (12) \end{aligned}$$

Hence, the function $t \rightarrow \int_{H \setminus \{0\}} |x|^\alpha e^{-t/|x|} M(dx) < \infty$ for almost all (Lebesgue measure) $t \in \mathbb{R}^+$. Its monotonicity gives that it is finite for all $t > 0$. Finally note that

$$e^{-1} \int_{\{|x|>1\}} |x|^\alpha M(dx) \leq \int_{\{|x|>1\}} |x|^\alpha e^{-1/|x|} M(dx) < \infty,$$

which completes the proof.

The converse part follows from the fact that $h_\alpha(|x|) \leq 2((2-\alpha)\alpha)^{-1}|x|^\alpha$ and Proposition 2. Thus the proof is complete.

3. Random integral representations. Random integral representation method allows to represent a random variable, more precisely its probability distribution, as a probability distribution of random integrals of the form $\int_{(a,b]} h(t) dY(r(t))$, where Y is the Lévy process (process with stationary independent increments, cadlag paths and starting from the origin), h is a real deterministic and r is deterministic and monotone with positive values (deterministic time change). A such method of a description of measures was introduced in Jurek-Vervaat (1983) for selfdecomposable measures; cf. also Jurek-Mason(1993), Chapter 3, Jurek (1982, 1985, 1988).

Theorem 1. *Let $\lambda(\cdot)$ be a Borel measure on positive half-line that is finite on sets bounded away from zero and let $\Lambda(t) := \lambda(\{s > 0 : s > t\})$, $t > 0$.*

Further, let $Y(t), t \geq 0$, be a cadlag Lévy process with values in a Hilbert space H and $Y(1)$, as infinitely divisible random element, is described by a triple $[a, R, M]$. Then in order that the limit

$$I_{(\alpha, \beta]} := \int_{(\alpha, \beta]} t dY(\Lambda(t)) \rightarrow \int_{(0, \infty)} t dY(\Lambda(t)) =: I_{(0, \infty)}, \quad (13)$$

exists in distribution, as $\alpha \downarrow 0$ and $\beta \uparrow \infty$, it is sufficient and necessary that

$$\begin{aligned} \int_0^\infty t \lambda(dt) < \infty, \text{ provided } a \neq 0; \int_0^\infty t^2 \lambda(dt) < \infty, \text{ provided } R \neq 0; \\ \int_{\{0 < \|x\| \leq 1\}} \|x\| \int_{\|x\|^{-1}}^\infty t \lambda(dt) M(dx) + \int_{\{\|x\| > 1\}} \|x\| \int_0^{\|x\|^{-1}} t \lambda(dt) M(dx) < \infty; \\ \text{and } M^{(\lambda)} \text{ is a Lévy spectral measure.} \end{aligned} \quad (14)$$

Furthermore, if the limit $I_{(0, \infty)}$ has representation $[a^{(\lambda)}, R^{(\lambda)}, M^{(\lambda)}]$ then

$$\begin{aligned} a^{(\lambda)} &= \left(\int_0^\infty t \lambda(dt) \right) \cdot a + \int_0^\infty \int_{H \setminus \{0\}} [1_B(tx) - 1_B(x)] t x M(dx) \lambda(dt); \\ R^{(\lambda)} &= \left(\int_0^\infty t^2 \lambda(dt) \right) \cdot R; \quad M^{(\lambda)}(A) = \int_0^\infty \int_{H \setminus \{0\}} 1_A(tx) M(dx) \lambda(dt). \end{aligned} \quad (15)$$

Proof. From the definition of random integrals $W_{(\alpha, \beta]} := \int_{(\alpha, \beta]} h(t) dY(\tau(t))$, where $h : (\alpha, \beta] \rightarrow \mathbb{R}$ and $\tau : (\alpha, \beta] \rightarrow \infty$ are deterministic functions, and τ is monotone one, and Y is a cadlag Lévy process, we have that

$$\mathbb{E}[e^{i \langle y, W_{(\alpha, \beta]} \rangle}] = \exp \int_{(\alpha, \beta]} \left(\log \mathbb{E}[e^{i \langle h(t) y, Y(1) \rangle}] \right) (\pm) d\tau(t), \quad (16)$$

where one takes the sign "+" for nondecreasing τ and "-" for nonincreasing τ ; cf. Jurek-Vervaat(1983) or Jurek-Mason (1993), Section 3.6. Hence using the Lévy-Khintchine formula (2) and taking $h(t) = t$ and $\tau = \Lambda$ in (16), we conclude that random element $I_{(\alpha, \beta]}$ has an infinitely divisible distribution

with the triple $[a_{(\alpha,\beta]}^{(\lambda)}, R_{(\alpha,\beta]}^{(\lambda)}, M_{(\alpha,\beta]}^{(\lambda)}]$ given as follows

$$\begin{aligned} a_{(\alpha,\beta]}^{(\lambda)} &= \left(\int_{(\alpha,\beta]} t \lambda(dt) \right) \cdot a + \int_{(\alpha,\beta]} t \int_{H \setminus \{0\}} [1_B(tx) - 1_B(x)] x M(dx) \lambda(dt); \\ R_{(\alpha,\beta]}^{(\lambda)} &= \left(\int_{(\alpha,\beta]} t^2 \lambda(dt) \right) \cdot R; \\ M_{(\alpha,\beta]}^{(\lambda)}(A) &= \int_{(\alpha,\beta]} \int_{H \setminus \{0\}} 1_A(tx) M(dx) \lambda(dt) = M^{(\lambda|_{(\alpha,\beta]})}(A), \end{aligned} \quad (17)$$

where the triple $[a, R, M]$ comes from the Lévy-Khintchine representation of the infinitely divisible random element $Y(1)$. As $\alpha \downarrow 0$ and $\beta \uparrow \infty$ then $M_{(\alpha,\beta]}^{(\lambda)} \uparrow M^{(\lambda)} \in \mathcal{M}(H)$, Gaussian covariance operators $R_{(\alpha,\beta]}^{(\lambda)} \rightarrow R^{(\lambda)}$. Finally, for the shift part note that

$$|[1_B(tx) - 1_B(x)]| = 1 \text{ iff } 1 < \|x\| \leq t^{-1} \text{ or } t^{-1} < \|x\| \leq 1,$$

and the second summand for a shift vector in (17) exits as a Bochner integral on the product space $(0, \infty) \times (H \setminus \{0\})$. Consequently, $I_{(\alpha,\beta]}$ converge in distribution to $I_{(0,\infty)}$ by Parthasarathy (1968), Theorem 5.5, because $M_{(\alpha,\beta]}^{(\lambda)} \uparrow M^{(\lambda)} \in \mathcal{M}(H)$. Thus the proof is complete.

REMARK 1. Since the random functions $\beta \rightarrow I_{(\alpha,\beta]}$ and $\alpha \rightarrow I_{(\alpha,\beta]}$ have independent increments (because so has Lévy process Y) we infer that all three modes of convergence: almost surely, in probability and in distribution) are equivalent; cf. Araujo-Gine (1980), Chapter 3, Theorem 2.10, p. 105.

As in previous papers Jurek&Vervaat (1983) or Jurek (1982, 1985, 1988) here we introduce the following *random integral mapping*

$$\mathcal{K}^{(\lambda)}(\mu) := \mathcal{L}\left(\int_0^\infty t dY_\mu(\Lambda(t))\right) \in ID \quad (18)$$

where $Y_\mu(t), t \geq 0$ is a cadlag Lévy process such that $\mathcal{L}(Y_\mu(1))a = \mu$ and $\Lambda(\cdot)$ is the cumulative distribution function or its tail function—note that from Proposition 2, λ , as a Lévy spectral measure, is finite on any half-line (a, ∞) , $a > 0$.

COROLLARY 3. *For probability measures of the form $\mathcal{K}^{(\lambda)}(\mu)$ one has*

$$\mathbf{E}[e^{i\langle y, \int_0^\infty t dY_\mu(\Lambda(t)) \rangle}] = \exp \int_0^\infty \log \mathbf{E}[e^{it\langle y, Y_\mu(1) \rangle}] \lambda(dt),$$

where $y \in E'$. Furthermore, the random integral mapping $\mathcal{K}^{(\lambda)}(\mu)$ has the following algebraic properties

$$\mathcal{K}^{(\lambda)}(\mu_1 \star \mu_2) = \mathcal{K}^{(\lambda)}(\mu_1) \star \mathcal{K}^{(\lambda)}(\mu_2), \quad \mathcal{K}^{(\lambda_1 + \lambda_2)}(\mu) = \mathcal{K}^{(\lambda_1)}(\mu) \star \mathcal{K}^{(\lambda_2)}(\mu)$$

Proof. The first is consequence of the definition of random integrals; cf. for analogous results in Jurek and Vervaat (1983) or Jurek and Mason (1993), Lemma 3.6.4. The second equality is a consequence of the formula (16) when one takes integrals over positive half-line and the function $h(t) = t$.

One of the advantages of random integral representation is that it allows easily to incorporate space and time changes. Here is an example.

COROLLARY 4. *For $a \in \mathbb{R}$, $c > 0$ and a random integral $\int_{(\alpha, \beta]} h(t) dY(\tau(t))$, where $h : (\alpha, \beta] \rightarrow \mathbb{R}$ and $\tau : (\alpha, \beta] \rightarrow \infty$ are deterministic functions, and τ is monotone one, and Y is a cadlag Lévy process, we have*

$$\begin{aligned} \left(\mathcal{L} \left(a \int_{(\alpha, \beta]} h(t) dY(\tau(t)) \right)^{\ast c} \right)^{\widehat{}}(y) &= \mathcal{L} \left(\int_{(\alpha, \beta]} a h(t) dY(c\tau(t)) \right)(y) = \\ &= \exp \int_{(\alpha, \beta]} \log \mathbf{E}[e^{i a h(t) \langle y, Y_{\mu}(c) \rangle}] d\tau(t), \quad y \in H. \end{aligned} \quad (19)$$

It is also true for integrals over half line, provided they exist.

Proof. Use (16) and the fact that $\mathcal{L}(Y(c)) = (\mathcal{L}(Y(1)))^{\ast c}$.

4. Two applications of the random integral method.

A). *Free infinite divisibility.* As in Section 2, let $e(dt)$ denotes the standard exponential distribution. Then from Example **A** we infer that $M^{(e)}$ is Lévy spectral measure (on H) whenever so is M . Furthermore by Theorem 1, formula (14), $R^{(e)} = 2R$, and

$$\begin{aligned} a^{(e)} &= a + \int_{\{\|x\| > 1\}} x(1 - e^{-\|x\|^{-1}}(1 + \|x\|^{-1}))M(dx) \\ &\quad + \int_{\{0 < \|x\| \leq 1\}} x e^{-\|x\|^{-1}}(1 + \|x\|^{-1})M(dx) \end{aligned} \quad (20)$$

exists in a Bochner sense. To this end note that $\lim_{s \rightarrow 0} s e^{-s^{-1}}(1 + s^{-1}) = 0$ and $\lim_{s \rightarrow \infty} s(1 - e^{-s^{-1}}(1 + s^{-1})) = 0$.

Furthermore, for $r > 0$ and a Borel D of the unit sphere $S = \{x : \|x\| = 1\}$, let us define *Lévy spectral function* $L_M(D; r)$ associated with the measure M as follows

$$L_M(D; r) = M(\{x : x \|x\|^{-1} \in D \text{ and } \|x\| > r\}). \quad (21)$$

Then using (7) we get

$$L_{M^{(e)}}(D; r) = \int_0^\infty L_M(D; rt^{-1}) e^{-t} dt = r \int_0^\infty L_M(D; s^{-1}) e^{-rs} ds, \quad r > 0. \quad (22)$$

Hence, $r^{-1}L_{M^{(e)}}(D; r), r > 0$, is a Laplace transform of (unique) function $L_M(D; s^{-1})$ and thus $M^{(e)}$ uniquely determines M . Hence, $a^{(e)}$ and M uniquely identifies a . All in all with Theorem 1 we conclude that

$$\mathcal{K}^{(e)} : ID \ni \mu \rightarrow \mathcal{L}\left(\int_0^\infty t dY_\mu(1 - e^{-t})\right) \in \mathcal{E} := \mathcal{K}^{(e)}(ID)$$

is well defined one-to-one random integral mapping, (23)

where $Y_\mu(t), t \geq 0$, is a cadlag Lévy process such that $\mathcal{L}(Y_\mu(1)) = \mu$. Consequently, we obtained convolution a subsemigroup $\mathcal{E} \subset ID$, which is characterized among infinitely divisible by the triples $[a^{(e)}, R^{(e)}, M^{(e)}]$ given by (15) and the kernel (2).

For a probability measure ν , let $\hat{\nu}$ denotes its Fourier transform (characteristic function). In terms of Fourier transforms elements representable as $\mathcal{K}^{(e)}(\cdot)$ are described as follows

COROLLARY 5. *In order for a function $g : H \rightarrow \mathbb{C}$ to be a characteristic function of a measure from the convolution semigroup \mathcal{E} it is necessary and sufficient that*

$$g(y) = \exp \left[i \langle y, a \rangle - \langle y, Ry \rangle + \int_{H \setminus \{0\}} \left(\frac{1}{1 - i \langle y, x \rangle} - 1 - i \langle y, x \rangle 1_{\{\|x\| \leq 1\}} \right) M(dx) \right], \quad (24)$$

where $a \in H$, R is non-negative, self-adjoint, trace operator and M is a Borel measure that integrates the function $\min(1, \|x\|^2)$ over H . In fact, g is a characteristic of the measure $\mathcal{K}^{(e)}([a, R, M])$.

One gets (24) by putting into (2) the triplet: the vector $a^{(e)}$, the covariance operator $R^{(e)}$ and the Lévy spectral measure $M^{(e)}$.

REMARK 2. One has two possibilities of looking at the class \mathcal{E} . Either, as a subset of ID with the triples $[a^{(e)}, R^{(e)}, M^{(e)}]$ and the kernel Φ from formula (2) or as a set of probability distributions given by triples $[a, R, M]$ and but with a new kernel

$$\begin{aligned} \Phi_1(y) := & [i < y, a > - < y, Ry > + \\ & \int_{H \setminus \{0\}} \left(\frac{1}{1 - i < y, x >} - 1 - i < y, x > 1_{\{\|x\| \leq 1\}} \right) M(dx)], \quad y \in H. \end{aligned} \quad (25)$$

Note that both kernels are additive in a, R and M , i.e., sums of those parameters correspond to convolution of probability measures.

(Compare similar comments in Jurek-Vervaat (1983) formula (4.3), pages 254-255, for the Lévy class L of selfdecomposable distributions.)

PROPOSITION 3. *Let $I^{(e)} := \int_0^\infty tdY(1-e^{-t})$ and $\phi_{I^{(e)}}(y)$, and $\phi_{Y(1)}(y)$, $y \in H$ are characteristic functions of $I^{(e)}$ and $Y(1)$, respectively. Then*

$$\begin{aligned} \log \phi_{I^{(e)}}(y) &= \int_0^\infty \log \phi_{Y(1)}(ty) e^{-t} dt, \\ \log \phi_{Y(1)}(y) &= \mathfrak{L}^{-1}[s^{-1} \log \phi_{I^{(e)}}(s^{-1}y; x)]|_{x=1}, \end{aligned} \quad (26)$$

where for each $y \in H$, \mathfrak{L}^{-1} is the inverse of the Laplace transform of the function $s^{-1} \log \phi_{I^{(e)}}(s^{-1}y)$. Hence, the mapping $\mathcal{K}^{(e)} : ID(H) \rightarrow \mathcal{E}$ is an algebraic isomorphism between convolution semigroups and for its inverse $(\mathcal{K}^{(e)})^{-1}$ we have

$$((\mathcal{K}^{(e)})^{-1}(\rho))^\wedge(y) = \exp \mathfrak{L}^{-1}[s^{-1} \log \hat{\rho}(s^{-1}y; x)]|_{x=1}, \quad y \in H.$$

Proof. From Corollary 3 we have

$$\log \phi_{I^{(e)}}(y) = \int_0^\infty \log \phi_{Y(1)}(ty) e^{-t} dt.$$

Putting, for each fixed $y \in H$,

$$f_y(s) := \log \phi_{I^{(e)}}(sy) \text{ and } g_y(s) := \log \phi_{Y(1)}(sy), \text{ for } s \in \mathbb{R},$$

and using the above relation we get

$$f_y(s) = \int_0^\infty g_y(ts)e^{-t}dt, \quad f_y(s^{-1}) = s \int_0^\infty g_y(x)e^{-sx}dx, \quad s > 0.$$

Consequently, $\frac{1}{s} f_y(\frac{1}{s}) = \mathfrak{L}[(g_y(x); s)]$ is the Laplace transform evaluated at s . This completes the proof.

In an abstract semigroup (\mathcal{G}, \circ) and element $g \in \mathcal{G}$ is said to be *infinitely divisible* if for each natural $n \geq 2$ there exists $g_n \in \mathcal{G}$ such that n -times operation $g_n \circ g_n \circ \dots \circ g_n = g$; cf. Hilgert, Hoffman, Lawson (1989). By $ID(\mathcal{G}) = (ID(\mathcal{G}), \circ)$ we denote a set of all \circ -infinite divisibility elements.

D. Voiculescu and others developed a theory of "a free probability". For our needs here let us recall briefly that with a probability measure μ , on a real line, one associates a complex valued function $V_\mu(z) := F_\mu^{-1}(z) - z, z \in \mathcal{D}$, where \mathcal{D} is an appropriately selected domain in upper complex half-plane and $F_\mu(z) := 1/G_\mu(z)$, where

$$G_\mu(z) := \int_{-\infty}^\infty \frac{1}{z-t} \mu(dt) \quad \text{is called the Cauchy transform.} \quad (27)$$

For two probability measures, on the real line, μ and ν , if the sum $V_\mu(z) + V_\nu(z)$ corresponds to another probability measure then we denote it by $\mu \square \nu$. Hence one can introduce \square -infinite divisibility and a semigroup $(ID(\mathcal{P}(\mathbb{R}), \square)$. From Bercovici and Pata (1999) and Barndorff-Nielsen and Thorbjørnsen (2002) we have that

$$\nu \in ID(\mathcal{P}(\mathbb{R}), \square) \quad \text{iff} \quad z V_\nu(z^{-1}) = az + \sigma^2 z^2 + \int_{\mathbb{R} \setminus \{0\}} \left(\frac{1}{1-zx} - 1 - zx1_{\{|x| \leq 1\}} \right) M(dx), \quad z \in \mathbb{C}^-, \quad (28)$$

where $a, \sigma \in \mathbb{R}$, and M integrates $\min(1, |x|^2)$ over \mathbb{R} , i.e., M is a Lévy spectral measure on real line.

COROLLARY 6. *A probability measure ν , on \mathbb{R} , is \square -infinitely divisible if and only if there exist a unique reals a and σ^2 and a Lévy spectral measure M such that*

$$(it) V_\nu((it)^{-1}) = \log(\mathcal{K}^{(e)}(\mu))^\wedge(t) = \log \left(\mathcal{L} \left(\int_0^\infty s dY_\mu(1 - e^{-s}) \right) \right)^\wedge(t), \quad t \in \mathbb{R},$$

(29)

where $(Y_\mu(t), t \geq 0)$ is a Lévy process such that $\mathcal{L}(Y_\mu(1)) = \mu = [a, \sigma^2, M]$. In other words, functions $t \rightarrow e^{itV_\nu((it)^{-1})}, t \in \mathbb{R}$, are characteristic functions and a class of measures corresponding to them coincides with the class \mathcal{E} . Furthermore, for the Voiculescu transform we have $V_\nu(it) = it \log(\mathcal{K}^{(e)}(\mu))^\sim(-t^{-1}), t \in \mathbb{R}$.

Proof. Use Corollary 3 for $E = \mathbb{R}$ and then apply Theorem 1 with the formula (18).

REMARK 3. Since both the kernel Φ_1 given by (25), (that appeared in the description of the class \mathcal{E}), and the kernel Φ given by (2), (that is the classical kernel from the Lévy-Khintchine), contain identical parameters a , R and M one has a natural identification between those convolution semigroups. (See Remark 2). Sometimes it is called Bercovici-Pata bijection between free and classical infinite divisibility. In the approach presented here we have explicitly constructed the isomorphism $\mathcal{K}^{(e)}$ between the semigroups \mathcal{E} and ID .

For the formula (24) in Corollary 5, or more precisely for the existence of Lévy process Y , it is necessary that μ is $*$ -infinitely divisible, while the Cauchy transform G_μ , given by (27), is defined for any (finite) measure. Here is a way of avoiding that difficulty. For any finite measure m , on a Hilbert or Banach space, let $e(m)$ denotes the compound Poisson measure. Since it is $*$ -infinitely divisible, (i.e., $e(m) \in ID(H)$), we can insert it into a Lévy (compound Poisson) process $Y_{e(m)}(t), t \geq 0$. Consequently,

$$\begin{aligned} \text{if } \mathbf{G}_m(y) &:= \int_H \frac{1}{1 - i \langle y, x \rangle} m(dx), \quad y \in H, \quad \text{then} \\ \mathbf{F}_m(y) &:= \log(\mathcal{K}^{(e)}(e(m)))^\sim(y) = \mathbf{G}_m(y) - m(H) = \int_H \frac{i \langle y, x \rangle}{1 - i \langle y, x \rangle} m(dx). \end{aligned} \tag{30}$$

To see that equalities recall that $e(m)^\sim(y) = \exp(\hat{m}(y) - m(H))$ and this with (18) and Corollary 3 give the above formula.

B). *Tempered stable probability measures.* Let us consider the example **B** from Section 2 for measures $\rho_\alpha(dt) = t^{-\alpha-1}e^{-t}dt$ on positive half-line with $0 < \alpha < 2$. Let us assume that

$$\int_H \|x\|^\alpha M(dx) < \infty \quad \text{and } M \text{ is a Borel measure on } H.$$

In the sequel, by ID_α we denote those infinitely divisible whose Lévy spectral measures satisfy the above integrability condition. Consequently, from Corollary 2 we have that both $M^{(\rho_\alpha)}$ and M are Lévy spectral measures and from Theorem 1 we get $R^{(\rho_\alpha)} = \Gamma(2 - \alpha)R$ is covariance operator of Gaussian measure. For the shift vector $a^{(\rho_\alpha)}$, we need three integrals; cf. formula (14). Firstly, note that

$$\begin{aligned} & \int_{\{0 < \|x\| \leq 1\}} \|x\| \int_{\|x\|^{-1}}^{\infty} t^{-\alpha} e^{-t} dt M(dx) \leq \\ & \int_{\{0 < \|x\| \leq 1\}} \|x\|^2 M(dx) \int_1^{\infty} t^{(2-\alpha)-1} e^{-t} dt M(dx) < \infty, \text{ for } 0 < \alpha < 2. \end{aligned}$$

And secondly, note that

$$\begin{aligned} & \int_{\{\|x\| > 1\}} \|x\| \int_0^{\|x\|^{-1}} t^{-\alpha} e^{-t} dt M(dx) = \int_{\{\|x\| > 1\}} \|x\| \gamma(1 - \alpha, \|x\|^{-1}) M(dx) \\ & = \int_{\{\|x\| > 1\}} \|x\|^\alpha \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\alpha + n)} \cdot \frac{1}{\|x\|^n} \right) M(dx) < \infty, \text{ for } 0 < \alpha < 1. \end{aligned}$$

Consequently, for $0 < \alpha < 1$, the random integral mapping

$$\mathcal{K}^{(\rho_\alpha)} : ID_\alpha \ni \mu \rightarrow \mathcal{L}\left(\int_0^\infty t dY_\mu(\Gamma(\alpha, t))\right) \in \mathcal{TS}_\alpha := \mathcal{K}^{(\rho_\alpha)}(ID_\alpha) \quad (31)$$

is well defined. In above $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ denote the incomplete Euler's gamma functions, i.e.,

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \quad x > 0, (\Re \alpha > 0); \quad \Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt, \quad x > 0.$$

Following Rosiński(2002) measures from the class \mathcal{TS}_α are called *tempered stable distributions*. In fact, they were introduced as infinitely divisible measures (on \mathbb{R}^d) without Gaussian parts with spectral measures of the form $M^{(\rho_\alpha)}$ from Example **B** in Section 2. [In Rosiński (2004), $M^{(\rho_\alpha)}$ appears as Lemma 2.2.] Let us mention here that tempered stable processes are of importance in statistical physics as they exhibits different local and global behavior; (cf. Mantegna and Stanley (1994), Novikov (1995), Kaponen (1995); comp. Corollary 7 below. However, our point of interest is that the tempered stable probability measures admit random integral representation as well.

PROPOSITION 4. Assume that $0 < \alpha < 1$. Let $I^{(\rho_\alpha)} := \int_0^\infty t dY(\Gamma(-\alpha, t))$ and $\phi_{I^{(\rho_\alpha)}}(y)$, and $\phi_{Y(1)}(y)$, $y \in H$, are characteristic functions of $I^{(\rho_\alpha)}$ and $Y(1)$, respectively. Then

$$\begin{aligned} \log \phi_{I^{(\rho_\alpha)}}(y) &= \int_0^\infty \log \phi_{Y(1)}(ty) t^{-\alpha-1} e^{-t} dt, \\ \log \phi_{Y(1)}(y) &= \mathfrak{L}^{-1}[s^\alpha \log \phi_{I^{(\rho_\alpha)}}(s^{-1}y; x)]|_{x=1}, \end{aligned} \quad (32)$$

where for each $y \in H$, \mathfrak{L}^{-1} is the inverse of the Laplace transform

$s^\alpha \log \phi_{I^{(\rho_\alpha)}}(s^{-1}y)$. Hence, the mapping $\mathcal{K}^{(\rho_\alpha)} : ID(H_\alpha) \rightarrow \mathcal{TS}_\alpha$ is an algebraic isomorphism between convolution semigroups. For its inverse $(\mathcal{K}^{(\rho_\alpha)})^{-1}$ we have

$$((\mathcal{K}^{(\rho_\alpha)})^{-1}(\rho))^\wedge(y) = \exp \mathfrak{L}^{-1}[s^\alpha \log \hat{\nu}(s^{-1}y; x)]|_{x=1}, \quad y \in H, \quad (\nu \in \mathcal{TS}_\alpha).$$

Proof is analogous to that of Proposition 3.

REMARK 4. The previous result is also true for $1 \leq \alpha < 2$ if one considers only Lévy processes with symmetric spectral measures M and zero shifts a .

Here is an example of usefulness of the random integral representation of random variables. The result below is announced in Rosiński (2002); also cf. Rosiński (2004). The proof below is based on the random integral representation of tempered stable probability measures.

COROLLARY 7. Let $0 < \alpha < 1$ and $X := \int_0^\infty t dY(\Gamma(-\alpha, t))$ be \mathbb{R}^d -valued random vector with $\mathcal{L}(Y(1)) = [0, 0, M] \in ID_\alpha$. Then

$$(\mathcal{L}(s^{-1/\alpha} X)^{*s}) \Rightarrow \eta_\alpha, \text{ as } s \rightarrow 0,$$

where η_α denotes the strictly stable law with exponent α

Proof. Using Corollary 4 with $a = s^{-1/\alpha}$ and $c = s$ we have

$$\begin{aligned} ((\mathcal{L}(\frac{1}{s^{1/\alpha}} X)^{*s}))^\wedge(y) &= \exp \int_0^\infty s \log \mathbf{E}[\exp i t / s^{1/\alpha} < y, Y(1) >] t^{-\alpha-1} e^{-t} dt \\ &= \exp \int_0^\infty \log \mathbf{E}[\exp i u < y, Y(1) >] u^{-\alpha-1} e^{-u s^{1/\alpha}} du \quad (\text{as } s \rightarrow 0) \\ &\rightarrow \exp \int_{\mathbb{R}^d \setminus \{0\}} \int_0^\infty [e^{iu \langle y, x \rangle} - 1 - i u \langle y, x \rangle 1_{\|x\| \leq 1}(x)] u^{-\alpha-1} du M(dx) = \\ &\exp[-c_\alpha \int_{\mathbb{R}^d \setminus \{0\}} |\langle y, x \rangle|^\alpha (1 - i \tan(\pi\alpha/2) \text{sign} \langle y, x \rangle) M(dx)], \end{aligned}$$

where $c_\alpha > 0$. (The last equality is obtained via contour integration; see any book on stable laws.) Finally, the last formula is the characteristic function of a strictly stable probability measure on \mathbb{R}^d .

References

- [1] Araujo, A. and Giné, E. (1980), *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York.
- [2] Barndorff-Nielsen, O. E. and Thorbjørnsen, S. (2002), Selfdecomposable and Lévy processes in free probability, *Bernoulli* **8**, 323–366.
- [3] Bercovici, H. and Voiculescu, D. V. (1993), Free convolution of measures with unbounded support, *Indiana Math. J.* **42**, 733–773.
- [4] Bercovici, H. and Pata, V. (1999), Stable laws and domains of attraction in free probability theory, *Ann. Math.* **149**, 1023–1060.
- [5] Hilgert, J., Hofmann, K. H. and J. D. Lawson, (1989), *Lie groups, convex cones and semigroups*. Oxford University Press, New York.
- [6] Iksanov, A. M., Jurek, Z. J. and Schreiber, B. M. (2004), A new factorization property of the selfdecomposable probability measures, *Ann. Probab.* **32**, No 2, 1356–1369.
- [7] Jurek, Z. J. (1982), An integral representation of operator-selfdecomposable random variables, *Bull. Acad. Polon. Sci., Sér. Sci. Math.* **30**, 385–393.
- [8] Jurek, Z. J. (1983), The classes $L_m(Q)$ of probability measures on Banach spaces, *Bull. Acad. Polon. Sci., Sér. Sci. Math.* **31**, 51–62.
- [9] Jurek, Z. J. (1985), Relations between the s -selfdecomposable and selfdecomposable measures, *Ann. Probab.* **13**, 592–608.
- [10] Jurek, Z. J. (1988), Random integral representations for classes of limit distributions similar to Levy class L_0 , *Probab. Th. Fields* **78**, 473–490.
- [11] Jurek, Z. J. (1990), On Lévy (spectral) measures of integral form on Banach spaces, *Probab. Math. Stat.* **11**, 139–148.

- [12] Jurek, Z. J. and Mason, J. D. (1993), *Operator-limit Distributions in Probability Theory*, Wiley Series in Probability and Mathematical Statistics, New York.
- [13] Jurek, Z. J. and Vervaat, W. (1983), An integral representation for self-decomposable Banach space valued random variables, *Z. Wahrsch. verw. Gebiete* **62**, 247–262.
- [14] Kaponen, I. (1995), Infinitely divisible distributions in turbulence, *Phys. Rev. E*, **52**, No 1, 1197–1199.
- [15] Mantegna, R. N. and Stanley, H. E. (1994), Stochastic processes with ultraslow convergence to a Gaussian: The truncated Lévy flight, *Phys. Rev. Letters* **73**, no. 22, 2946–2949.
- [16] Novikov, E. A. (1994), Infinitely divisible distributions in turbulence, *Phys. Rev. E*, **50**, No 5, 3303–3305.
- [17] O'Connor, T. A. (1979), Infinitely divisible distributions similar to class L , *Z. Wahrsch. verw. Gebiete* **50**, 265–271.
- [18] Parthasarathy, K.R. (1968), *Probability measures on metric spaces*, Academic Press, New York and London, 1968.
- [19] Rosiński, J. (2002), Tempered stable processes, *2nd MaPhySto Lévy Conference*, Aarhus University. <http://maphysto.dk/publications/MPS-misc/2002/22.pdf>, pp.215-220.
- [20] Rosiński, J. (2004), Tempering stable processes; Preprint.

Institute of Mathematics, University of Wrocław, 50-384 Wrocław, Poland.
[E-mail: zjjurek@math.uni.wroc.pl]